# On the classification of Kontsevich formality maps

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based on the joint paper with Assar Andesson, arXiv:1911.09089

## 1. Formal Poisson structures

Let $V$ be a finite-dimensional $\mathbb{Z}$-graded vector space over a field $\mathbb{K}$ of characteristic zero (say, $V = \mathbb{R}^d$, $\mathbb{K} = \mathbb{R}$) and $V^* := \mathrm{Hom}(V, \mathbb{K})$ its dual.

$\mathcal{O}_{\mathcal{M}} := \widehat{\odot^\bullet} V$ is the *algebra of formal smooth functions* on the dual vector space $V^*$ understood as a formal manifold $\mathcal{M}$.

The *Lie algebra of smooth vector fields* = the Lie algebra of derivations of $\mathcal{O}_{\mathcal{M}}$,

$$\mathcal{T}\mathcal{M} := \mathrm{Der}(\mathcal{O}_V) \simeq \mathrm{Hom}(V, \widehat{\odot^\bullet} V) \simeq \prod_{m \geq 0} \mathrm{Hom}(V, \odot^m V),$$

The Lie algebra of polyvector fields:

$$\mathcal{T}_{poly}\mathcal{M} := \wedge^\bullet T_{\mathcal{M}} \simeq \prod_{m\geq 0, n\geq 0} \mathrm{Hom}(\wedge^n V, \odot^m V)[-n] = \prod_{k \geq 0} \odot^k (V^*[-1] \oplus V)$$

A *formal graded Poisson structure* on $\mathcal{M}$ is a degree 2 element $\pi \in \mathcal{T}_{poly}\mathcal{M}$

$$\pi = \sum_{n,m=0}^{\infty} \pi_n^m, \qquad \pi_n^m \in \mathrm{Hom}(\wedge^n V, \odot^m V)[2-n]$$

such that

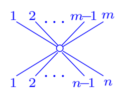
$$[\pi, \pi] = 0,$$

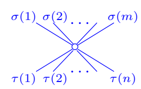
## 2. From formal Poisson structures to wheeled props

Interpret each homogeneous monomial $\pi_n^m$ as an equivariant morphism of $\mathbb{S}^m \times \mathbb{S}^n$-modules:

$$\pi_n^m : sgn_n \otimes \mathbf{1}_m[n-2] \longrightarrow \mathrm{Hom}(\otimes^n V, \otimes^m V)$$

The basis element of $sgn_n \otimes \mathbf{1}_m[n-2]$ is denoted by a graph with one vertex and $n$ in-legs and $m$ out-legs (called $(m,n)$-*corolla*, a flow is assumed to run from the bottom to the top),

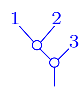


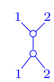
$= sgn(\tau) \cdot$


$\forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n$

Why such a notation? Linear maps in $\mathrm{Hom}(\otimes^\bullet V, \otimes^\bullet V)$ can be composed = out-legs can be glued to in-legs.

Given a collection of maps $\{\pi_n^m\}$ as above, one can uniquely maps like these ones



$\longrightarrow \mathrm{Hom}(V, \otimes^3 V)$,


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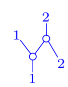
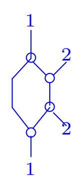
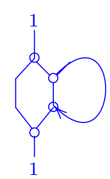


$\longrightarrow \mathrm{Hom}(\otimes^2 V, \otimes^2 V)$

or

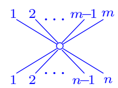


$\longrightarrow \mathrm{Hom}(\otimes^2 V, \otimes^2 V)$,


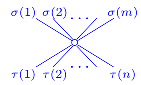
$\in \mathrm{Hom}(V, V)$

One must be careful: if $V$ is *infinite*-dimensional, then the graph with closed path of directed edges called *wheels* (such as the second one just above), must be *prohibited*.

Let $\mathcal{H}olieb^{\star}_{1,0}$ be the vector space of all possible graphs obtained by gluing out-legs of the generating corollas to




$= sgn(\tau) \cdot$



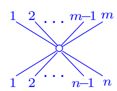

$\forall \sigma \in \mathbb{S}_m, \forall \tau \in \mathbb{S}_n$

in-legs of (other) generating corollas *with no wheels*.

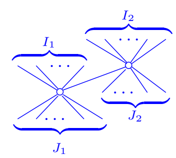
Let $\mathcal{H}olieb^{\star\circlearrowright}_{1,0}$ be the vector space of all possible graphs obtained by gluing out-legs of the generating corollas
to in-legs of (other) generating corollas *with possibly wheels*.

Both $\mathcal{H}olieb^{\star}_{1,0}$ and $\mathcal{H}olieb^{\star\circlearrowright}_{1,0}$ can be made into *complexes* by setting the differential to be given by its value on the generators as

$\delta$

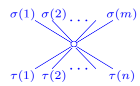



$$= \sum_{\substack{[1,\ldots,m]=I_1\sqcup I_2 \\ |I_1|\geq 0, |I_2|\geq 1}} \sum_{\substack{[1,\ldots,n]=J_1\sqcup J_2 \\ |J_1|\geq 1, |J_2|\geq 1}} \pm$$

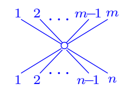



It takes care about the Schouten bracket in $\mathcal{T}_{poly}(\mathcal{M})$!

One can define similarly $\mathcal{H}olieb^{\star}_{c,d}$ and $\mathcal{H}olieb^{\star\circlearrowright}_{c,d}$ using the following generating corollas in degree $1 + c(1-m) + d(1-n)$




$$= (-1)^{c|\sigma|+d|\tau|}$$




The superscript $\star$ in the notation indicates that we consider in this paper an *extended version* of the (unwheeled) family of props $\mathcal{H}olieb_{c,d}$ studied earlier by [SM and Thomas Willwacher 2015]. The prop $\mathcal{H}olieb_{c,d}$ controls a *truncated* version of the above formal power series

$$\pi = \sum_{\substack{n,m\geq 1\\ n+m\geq 3}}^{\infty} \pi_n^m, \qquad \pi_n^m \in \mathrm{Hom}(\wedge^n V, \odot^m V)[2-n]$$

Such a truncation makes perfect sense in the context of the theory of minimal resolutions of $(c,d)$ Lie bialgebras, the case $c = d = 1$ corresponding to ordinary Lie bialgebras.

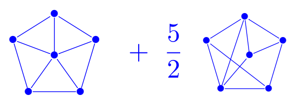
In [SM and Thomas Willwacher 2015] we computed the complex of derivations (infinitesimal deformations) of the prop $\mathcal{H}olieb_{c,d}$ and proved

$$H^\bullet(\mathrm{Der}(\mathcal{H}olieb_{c,d})) = H^\bullet(\mathsf{fGC}_{c+d})$$

where

$$\mathsf{fGC}_{c+d}^{\geq 2} := \widehat{\odot^\bullet}\left((\mathsf{GC}_{c+d}^{\geq 2} \oplus \mathbb{K})[-c-d]\right)[c+d],$$

and $\mathsf{GC}_{c+d}$ is the classical the Kontsevich graphs complex of connected graphs like this one



(with an empty graph added as a non-trivial element).

The case $\mathsf{fGC}_2$ (i.e. the case $c = d = 1$ of ordinary Lie bialgebras) is of special interest as Thomas Willwacher (2010) has proven

$$H^0(\mathsf{fGC}_2) = \mathfrak{grt}$$

where $\mathfrak{grt}$ is the Lie algebra of the famous Grothendieck-Teichmüller group $GRT$.

Conclusion from

$$H^\bullet(\mathrm{Der}(\mathcal{H}olieb_{c,d})) = H^\bullet(\mathsf{fGC}_{c+d})$$

[SM and Thomas Willwacher 2015] :

(i) The Grothendieck-Teichmüller group $GRT$ acts faithfully (and essentially transitively) on the completion of the properad $\mathcal{H}olieb_{1,1}$ governing ordinary Lie bialgebras, up to homotopy.

(ii) The same is true for *involutive* Lie bialgebras.

This result was used in [SM and Thomas Willwacher 2016] to classify all equivalence classes of deformation quantizations of (possibly *infinite*-dimensional) Lie bialgebras

$$\{\text{the set of such quantizations}\} = \{\text{the set } \mathcal{DA} \text{ of Drinfeld associators}\}.$$

What about $\mathcal{H}olieb^{\circlearrowright}_{1,1}$ and what about homotopy classes of deformation quantizations of only *finite*-dimensional Lie bialgebras. This is a much more difficult case comparing to the above story $\mathcal{H}olieb_{1,1}$.

Conjecture [SM]

$$\text{the set of such quantizations} = \mathcal{DA} \times \mathcal{DA} \simeq \mathcal{DA} \times GRT$$

What it means: in finite dimensions the dualization endofunctor which exchanges Lie and co-Lie (resp. Ass and co-Ass structures) structures does *not* commute with the deformation quantization functor even up to homotopy equivalence. The difference is controlled by the second factor.

The first main purpose of the present work is the study of the deformation theory of the *full* prop $\mathcal{H}olieb_{c,d}^{\star\circlearrowright}$ (in fact of their completed versions) which, for $c=0, d=1$ controls finite-dimensional Poisson structures. This is much easier that the case $\mathcal{H}olieb_{c,d}$ considered earlier!

THEOREM [Assar Andersson and SM, 2019] *There is an isomorphism of Lie algebras*

$$H^\bullet(\mathrm{Der}(\mathcal{H}olieb_{c,d}^{\star\circlearrowright})) = H^\bullet(\mathsf{fGC}_{c+d})$$

*In particular,*

$$H^0(\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star\circlearrowright})) = \mathfrak{grt}$$

By contrast

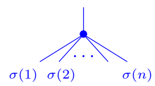
$$H^0(\mathrm{Der}(\mathcal{H}olieb_{0,1}^{\star})) = 0$$

that is, the completion of the prop $\mathcal{H}olieb_{1,0}^{\star}$ governing *infinite*-dimensional formal Poisson structures admits no homotopy non-trivial derivations (up to rescalings).

## 3. Homotopy classification of M. Kontsevich formality maps

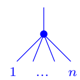
The operad of strongly homotopy *curved* associative algebras $c\mathcal{A}ss_\infty$ is generated by the following family of planar corollas in degree $2-n$

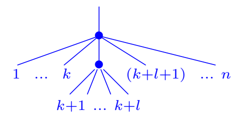
$$\left( \quad \right)_{\sigma\in\mathbb{S}_n}, \quad \forall\, n \geq 0$$




It is equipped with the differential given on the generators by

$$\delta \quad = \sum_{k=0}^{n}\sum_{l=0}^{n-k}(-1)^{k+l(n-k-l)+1} \quad .$$




Maxim Kontsevich (1997) formality map associates

$$\{\text{any formal Poisson structure } \pi \text{ on } V^*\} \Rightarrow \{\mathcal{A}ss_\infty\text{-structure on } \widehat{\odot^\bullet} V\}$$

given in terms of polydifferential operators constructed from $\pi$.

In our approach:
the l.h.s. is a representation of $\mathcal{H}olieb_{1,0}^{\star\circlearrowright}$ in $V$,
the r.h.s. is a representation of $c\mathcal{A}ss_\infty$ in $\widehat{\odot^\bullet} V$,

There is an exact polydifferential functor constructed in [SM and Thomas Willwacher 2015, a paper on ribbon graphs]

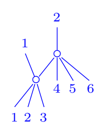
$$\mathcal{O} : \text{Category of dg props} \longrightarrow \text{Category of dg operads}$$

which has the property:

given any dg wheeled prop $\mathcal{P}$ and its representation in a vector space $V$
there is a canonically associated
dg operad $\mathcal{O}(\mathcal{P})$ and its representation in $\widehat{\odot^\bullet} V$.

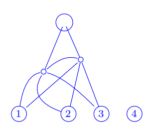
Applying $\mathcal{O}$ to $\mathcal{H}olieb_{1,0}^{\star\circlearrowright}$ we get a dg operad $\mathcal{O}(\mathcal{H}olieb_{1,0}^{\star\circlearrowright})$. The functor acts explicitly as follows:

Given, say, an element

$e =$  $\in \widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright}$



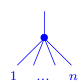
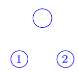
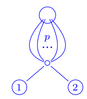
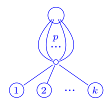
it can generate an element

$\hat{e} =$  $\in \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright})$

Maxim Kontsevich (1997) universal formality map (or any other universal formality map) can be understood now as a morphism of dg operads

$$\mathcal{F} : c\mathcal{A}ss_\infty \longrightarrow \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1}).$$

satisfying the following non-triviality condition

$\mathcal{F}\Big($  $\Big) = \begin{cases}$  $+ \sum_{p\geq 0} \frac{1}{p!}$  $+ O(2)$ if $n=2$ \\ $\sum_{p\geq 0} \frac{1}{p!}$  $+ O(2)$ otherwise $\end{cases}$

where the summations $\sum_{p\geq 0}$ run over the number of edges connecting the internal vertex to the external out-vertex and $O(2)$ stands for the space graphs with the number of *internal vertices* $\geq 2$.

To classify M.Konstevich formality maps one has to compute cohomology of

$$\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}^{\star\circlearrowleft}_{0,1})\right)$$

which controls the deformation theory of any formality map $\mathcal{F}$ (see e.g. SM and Bruno Vallette (2007) for a generic construction of such deformation complexes).

Our second main result is the computation of its cohomology in terms of the M. Kontsevich graph complex $\mathsf{fGC}_2^{\geq 2}$.

**Theorem** [Assar Andersson and SM (2019)] *Then there is a canonical morphism of complexes*

$$\mathsf{fGC}_2 \longrightarrow \mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\mathcal{H}olieb_{0,1}^{\star\circlearrowright})\right)[1]$$

*which is a quasi-isomorphism.*

Proof is very short: it uses contractibility of the permutahedra cell complexes.

Hence

$$H^{i+1}\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright})\right)\right) = H^i(\mathsf{fGC}_2)$$

which in the special case $i = 0$ reads as

$$H^1\left(\mathsf{Def}\left(c\mathcal{A}ss_\infty \xrightarrow{\mathcal{F}} \mathcal{O}(\widehat{\mathcal{H}olieb}_{0,1}^{\star\circlearrowright})\right)\right) = H^0(\mathsf{fGC}_2) = \mathfrak{grt}$$

and hence gives us a new (very short) proof of the following remarkable Theorem by V. Dolgushev.

**Theorem** [Vasily Dolgushev (2011)] *The Grothendieck-Teichmüller group $GRT$ acts freely and transitively on the set of homotopy classes of universal Kontsevich formality morphisms.*

This Theorem implies the identification of the set of homotopy classes of formality maps with the set of V. Drinfeld associators.

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